THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Suggested Solution to Assignment 2

1. The given geometry is not an incidence geometry since there exist no three noncollinear points, which violates axiom **I3**.

   (Remark: However, this geometry satisfies axioms **I1** and **I2**.)

2. (a) By axiom **I2**, there exist two points on the line l.

   Suppose the contrary, there exists no point P such that P does not lie on l, i.e. for all point P, P lies on l. By **I1**, l is the only line in the geometry.

   Then, there exist no distinct three noncollinear points, which violates axiom **I3**.

   (b) By axiom **I3**, there exist three noncollinear points R, S and T.

   (Case 1) $P \in \{R, S, T\}$

   Without loss of generality, let $R = P$.

   By axiom **I1**, there exist unique lines l_{PS} and l_{PT} such that $P, S \in l_{PS}$ and $P, T \in l_{PT}$.

   l_{PS} and l_{PT} must be distinct lines, otherwise it contradicts to the assumption that P, S and T are noncollinear.

   Then l_{PS} and l_{PT} are the distinct lines required.

   (Case 2) $P \notin \{R, S, T\}$

   By axiom **I1**, there exists unique line l_{RS} such that $R, S \in l_{RS}$.

   If $P \in l_{RS}$. By axiom **I1**, there exists unique lines l_{PT} such that $P, T \in l_{PT}$.

   l_{RS} and l_{PT} must be distinct lines, otherwise it contradicts to the assumption that P, S and T are noncollinear.

   Then l_{RS} and l_{PT} are the distinct lines required.

   If $P \notin l_{RS}$. There exist unique lines l_{PR} and l_{PS} such that $P, R \in l_{PR}$ and $P, S \in l_{PS}$.

   l_{PR} and l_{PS} must be distinct lines. Otherwise the line $L = l_{PR} = l_{PS}$ passes through R and S which forces $L = l_{RS}$ by axiom **I1**, and so P lies on $L = l_{RS}$ which is a contradiction.

   Then l_{PR} and l_{PS} are the distinct lines required.

   (c) By using the result in (a), there exists a point P such that $P \notin l$.

   Also, by axiom **I2**, l contains two distinct points Q and R.

   By axiom **I1**, there exist unique lines l_{PQ} and l_{PS} such that $P, Q \in l_{PQ}$ and $P, R \in l_{PR}$.

   l_{PQ} and l_{PR} must be distinct lines. Otherwise the line $L = l_{PQ} = l_{PR}$ passes through Q and R which forces $L = l_{QR} = l$ by axiom **I1**, and so P lies on $L = l_{QR}$ which is a contradiction.

   Then l_{PQ} and l_{PR} are the distinct lines required.

3. We have to verify the Klein Disk satisfy the following axioms:

   I1. Let P and Q be two distinct points in the Klein Disk.

   (Existence) Consider the ordinary straight line L_{PQ} which passes throguh P and Q in $\mathbb{R}^2$ and let l_{PQ} be the intersection of the Klein Disk and L_{PQ}. By definition, l_{PQ} is a line which passes throguh P and Q.

(Uniqueness) Suppose that l is a line which passes throguh P and Q.

By definition, l must be the intersection of the Klein Disk and some straight line L in $\mathbb{R}^2$.

Therefore, P and Q lies on L and by the uniqueness of straight line in $\mathbb{R}^2$, $L = L_{PQ}$.

As a result, $l = l_{PQ}$.

I2. By definition, every line in the Klein Disk is an open line segment in $\mathbb{R}^2$ which contains at least two points.

I3. Choose three distinct points $(0,0)$, $(0,1/2)$ and $(1/2,0)$ in the Klein Disk.

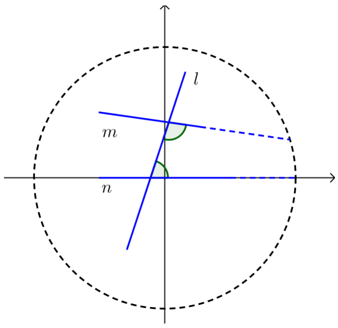
Suppose that they are collinear, then there exists a line l in the Klein Disk such that these three points lies on l.

However, by definition, l is the intersection of a straight line L in $\mathbb{R}^2$ and the Klein Disk, which means $(0,0)$, $(0,1/2)$ and $(1/2,0)$ are collinear points in $\mathbb{R}^2$ which is a contradiction.

Therefore, $(0,0)$, $(0,1/2)$ and $(1/2,0)$ are three noncollinear points in the Klein Disk.

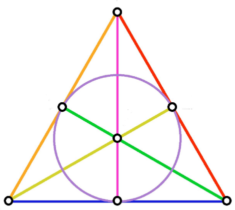
Therefore, the Klein Disk is an incidence geometry.

Pick two line segments m and n in the Klein Disk and a line segment l intersect m and n which forms two interior angles on the same side with the sum less than two right angles (see the figure below).



However, when we extend the line segments m and n, there is no intersection. Therefore, the Klein Disk does not satisfy Euclid's fifth axiom.

4. Consider the Fano plane:



Pick the line coloured in blue and pick the point in the center.

It can be seen that all lines (green, yellow and pink) passing through the chosen point has intersection with the blue one, so those lines are not parallel to the blue line by definition.

There exists no line that passes through the given point which is parallel to the blue line and the Fano plane does not satisfy **P**'.

5. Consider the following geometries

   (a) $\mathcal{S}_1 = \{A, B, C, D\}$ and $\mathcal{L}_1 = \{\{B, C, D\}, \{A, C, D\}, \{A, B, D\}, \{A, B, C\}\}$;

   (b) $\mathcal{S}_2 = \mathbb{R}^2$ and $\mathcal{L}_2 =$ straight lines in usual sense;

   (c) $\mathcal{S}_3 = \{A, B, C\}$ and $\mathcal{L}_3 = \{\{A, B\}, \{B, C\}, \{C, A\}\}$;

   (d) $\mathcal{S}_4 = \{A, B, C\}$ and $\mathcal{L}_4 = \{\{A, B, C\}\}$,

   where $\mathcal{S}_i$ and $\mathcal{L}_i$ are the set of points and lines of the geometry respectively, for $i = 1, 2, 3, 4$.

   Then, we can check that $(\mathcal{S}_i, \mathcal{L}_i)$ does not satisfy the $i-th$ axiom but satisfy all the other.